

THE ENTROPY OF RANDOM-FREE GRAPHONS AND PROPERTIES

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ABSTRACT. Every graphon defines a random graph on any given number n of vertices. It was known that the graphon is random-free if and only if the entropy of this random graph is subquadratic. We prove that for random-free graphons, this entropy can grow as fast as any subquadratic function. However, if the graphon belongs to the closure of a random-free graph property, then the entropy is $O(n \log n)$. We also give a simple construction of a non-stepfunction random-free graphon for which this entropy is linear, refuting a conjecture of Janson.

1. INTRODUCTION

In recent years a theory of convergent sequences of dense graphs has been developed. One can construct a limit object for such a sequence in the form of certain symmetric measurable functions called graphons. Every graphon defines a random graph on any given number of vertices. In [HJS] several facts about the asymptotics of the entropies of these random variables are established. These results provide a good understanding of the situation when the graphon is not “random-free”. However in the case of the random-free graphons they completely trivialize. The purpose of this article is to study these entropies in the case of the random-free graphons.

1.1. Preliminaries. For every natural number n , denote $[n] := \{1, \dots, n\}$. In this paper all graphs are simple and finite. For a graph G , let $V(G)$ and $E(G)$, respectively denote the set of the vertices and the edges of G . Let \mathcal{U} denote set of all graphs up to an isomorphism. Moreover, for $n \geq 0$, let $\mathcal{U}_n \subset \mathcal{U}$ denote the set of all graphs in \mathcal{U} with exactly n vertices. We will usually work with labeled graphs. For every $n \geq 1$, denote by \mathcal{L}_n the set of all graphs with vertex set $[n]$.

The *homomorphism density* of a graph H in a graph G , denoted by $t(H; G)$, is the probability that a random mapping $\phi : V(H) \rightarrow V(G)$ preserves adjacencies, i.e. $uv \in E(H)$ implies $\phi(u)\phi(v) \in E(G)$. The *induced density* of a graph H in a graph G , denoted by $p(H; G)$, is the probability that a random *embedding* of the vertices of H in the vertices of G is an embedding of H in G .

We call a sequence of finite graphs $(G_n)_{n=1}^\infty$ *convergent* if for every finite graph H , the sequence $\{p(H; G_n)\}_{n=1}^\infty$ converges. It is not difficult to construct convergent sequences $(G_n)_{n=1}^\infty$ such that their limits cannot be recognized as graphs, i.e. there is no graph G , with $\lim_{n \rightarrow \infty} p(H; G_n) = p(H; G)$ for every H . Thus naturally one considers $\overline{\mathcal{U}}$, the completion of \mathcal{U} under this notion of convergence. It is not hard to see that $\overline{\mathcal{U}}$ is a compact metrizable space which contains \mathcal{U} as a dense subset. The elements of the complement $\mathcal{U}^\infty := \overline{\mathcal{U}} \setminus \mathcal{U}$ are called *graph limits*. Note that a sequence of graphs $(G_n)_{n=1}^\infty$ converges to a graph limit Γ if and only if $|V(G_n)| \rightarrow \infty$ and $p(H; G_n) \rightarrow p(H; \Gamma)$ for every graph H . Moreover, a graph limit is uniquely determined by the numbers $p(H; \Gamma)$ for all $H \in \mathcal{U}$.

It is shown in [LS06] that every graph limit Γ can be represented by a *graphon*, which is a symmetric measurable function $W : [0, 1]^2 \rightarrow [0, 1]$. The set of all graphons are denoted by \mathcal{W}_0 . Given a graph G with vertex set $[n]$, we define the corresponding graphon $W_G : [0, 1]^2 \rightarrow \{0, 1\}$ as follows. Let $W_G(x, y) := A_G([xn], [yn])$ if $x, y \in (0, 1]$, and if $x = 0$ or $y = 0$, set W_G to 0. It is easy to see that if $(G_n)_{n=1}^\infty$ is a graph sequence that converges to a graph limit Γ , then for every graph H ,

$$p(H; \Gamma) = \lim_{n \rightarrow \infty} \mathbb{E} \left[\prod_{uv \in E(H)} W_{G_n}(x_u, x_v) \prod_{uv \in E(H)^c} (1 - W_{G_n}(x_u, x_v)) \right],$$

where $\{x_u\}_{u \in V(H)}$ are independent random variables taking values in $[0, 1]$ uniformly, and $E(H)^c = \{uv : u \neq v, uv \notin E(H)\}$. Lovász and Szegedy [LS06] showed that for every graph limit Γ , there exists a graphon

W such that for every graph H , we have $p(H; \Gamma) = p(H; W)$ where

$$p(H; W) := \mathbb{E} \left[\prod_{uv \in E(H)} W(x_u, x_v) \prod_{uv \in E(H)^c} (1 - W(x_u, x_v)) \right].$$

Furthermore, this graphon is unique in the following sense: If W_1 and W_2 are two different graphons representing the same graph limit, then there exists a measure-preserving map $\sigma : [0, 1] \rightarrow [0, 1]$ such that

$$W_1(x, y) = W_2(\sigma(x), \sigma(y)), \quad (1.1)$$

almost everywhere [BCL10]. With these considerations, sometimes we shall not distinguish between the graph limits and their corresponding graphons. We define the δ_1 distance of two graphons W_1 and W_2 as

$$\delta_1(W_1, W_2) = \inf \|W_1 - W_2 \circ \sigma\|_1,$$

where the infimum is over all measure-preserving maps $\sigma : [0, 1] \rightarrow [0, 1]$.

A graphon W is called a *stepfunction*, if there is a partition of $[0, 1]$ into a finite number of measurable sets S_1, \dots, S_n so that W is constant on every $S_i \times S_j$. The partition classes will be called the *steps* of W .

Let W be a graphon and $x_1, \dots, x_n \in [0, 1]$. The random graph $G(x_1, \dots, x_n, W) \in \mathcal{L}_n$ is obtained by including the edge ij with probability $W(x_i, x_j)$, independently for all pairs (i, j) with $1 \leq i < j \leq n$. By picking x_1, \dots, x_n independently and uniformly at random from $[0, 1]$, we obtain the random graph $G(n, W) \in \mathcal{L}_n$. Note that for every $H \in \mathcal{L}_n$,

$$\Pr[G(n, W) = H] = p(H; W).$$

1.2. Graph properties and Entropy. A subset of the set \mathcal{U} is called a *graph class*. Similarly a *graph property* is a property of graphs that is invariant under graph isomorphisms. There is an obvious one-to-one correspondence between graph classes and graph properties and we will not distinguish between a graph property and the corresponding class. Let $\mathcal{Q} \subseteq \mathcal{U}$ be a graph class. For every $n > 1$, we denote by \mathcal{Q}_n the set of graphs in \mathcal{Q} with exactly n vertices. We let $\overline{\mathcal{Q}} \subseteq \overline{\mathcal{U}}$ be the closure of \mathcal{Q} in $\overline{\mathcal{U}}$.

Define the *binary entropy* function $h : [0, 1] \mapsto \mathbb{R}_+$ as $h(x) = -x \log(x) - (1 - x) \log(1 - x)$ for $x \in (0, 1)$ and $h(0) = h(1) = 0$ so that h is continuous on $[0, 1]$ where here and throughout the paper $\log(\cdot)$ denotes the logarithm to the base 2. The *entropy* of a graphon W is defined as

$$\text{Ent}(W) := \int_0^1 \int_0^1 h(W(x, y)) dx dy.$$

Note that it follows from the uniqueness result (1.1) that entropy is a function of the underlying graph limit, and it does not depend on the choice of the graphon representing it. It is shown in [Ald85] and [Jan, Theorem D.5] that

$$\lim_{n \rightarrow \infty} \frac{\text{Ent}(G(n, W))}{\binom{n}{2}} = \text{Ent}(W). \quad (1.2)$$

A graphon is called *random-free* if it is $\{0, 1\}$ -valued almost everywhere. Note that a graphon W is random-free if and only if $\text{Ent}(W) = 0$, which by (1.2) is equivalent to $\text{Ent}(G(n, W)) = o(n^2)$. Our first theorem shows that this is sharp in the sense that the growth of $\text{Ent}(G(n, W))$ for random-free graphons W can be arbitrarily close to quadratic.

Theorem 1.1. *Let $\alpha : \mathbb{N} \rightarrow \mathbb{R}_+$ be a function with $\lim_{n \rightarrow \infty} \alpha(n) = 0$. Then there exists a random-free graphon W such that $\text{Ent}(G(n, W)) = \Omega(\alpha(n)n^2)$.*

A graph property \mathcal{Q} is called *random-free* if every $W \in \overline{\mathcal{Q}}$ is random-free. Our next theorem shows that in contrast to Theorem 1.1, when a graphon W is the limit of a sequence of graphs with a random-free property, then $\text{Ent}(G(n, W))$ cannot grow faster than $O(n \log n)$.

Theorem 1.2. *Let \mathcal{Q} be a random-free property, and let W be the limit of a sequence of graphs in \mathcal{Q} . Then $\text{Ent}(G(n, W)) = O(n \log n)$.*

Remark 1.3. We defined $G(n, W)$ as a *labeled* graph in \mathcal{L}_n . Both Theorems 1.1 and 1.2 remain valid if we consider the random variable $G_u(n, W)$ taking values in \mathcal{U}_n obtained from $G(n, W)$ by forgetting the labels. Indeed, $\text{Ent}(G_u(n, W)) = \text{Ent}(G(n, W)) - \text{Ent}(G(n, W) \mid G_u(n, W))$ and $\text{Ent}(G(n, W) \mid G_u(n, W) = H) = O(n \log n)$ for every $H \in \mathcal{U}_n$. It follows that

$$\text{Ent}(G(n, W)) - O(n \log n) \leq \text{Ent}(G_u(n, W)) \leq \text{Ent}(G(n, W)).$$

2. PROOF OF THEOREM 1.1

For every positive integer m , let F_m denote the unique bigraph $([m], [2^m], E)$ with the property that the vertices in $[2^m]$ all have different sets of neighbors. The *transversal-uniform* graph is the unique graph (up to an isomorphism) with vertex set \mathbb{N} which satisfies the following property. The vertices are partitioned into sets $\{A_i\}_{i=1}^\infty$ with $\log |A_i| = \sum_{j=1}^{i-1} |A_{j-1}|$. There are no edges inside A_i 's, and for every i , the bigraph induced by $(\cup_{j=1}^{i-1} A_j, A_i)$ is isomorphic to $F_{\sum_{j=1}^{i-1} |A_j|}$.

Let $\mathcal{I} = \{I_i\}_{i \in \mathbb{N}}$ be a partition of $[0, 1]$ into intervals. We define its corresponding *transversal-uniform* graphon $W_{\mathcal{I}}$ by assigning weights $|I_i|/|A_i|$ to all the vertices in A_i in the transversal-uniform graph G_U described above. More precisely, we partition each I_i into $|A_i|$ equal size intervals (corresponded with elements in A_i), and mapping all the points in each of these subintervals to its corresponding vertex in A_i . This measurable surjection $\pi_{\mathcal{I}} : [0, 1] \rightarrow \mathbb{N}$, together with the transversal-uniform graph described above defines the transversal-uniform graphon $W_{\mathcal{I}}$ by setting

$$W_{\mathcal{I}}(x, y) = \begin{cases} 1 & \text{if } \pi(x)\pi(y) \in E(G_U), \\ 0 & \text{if } \pi(x)\pi(y) \notin E(G_U). \end{cases}$$

Note that by construction $W_{\mathcal{I}}$ has the following property. Let $s < k$ be positive integers, and $x_1, \dots, x_s \in \cup_{i < k} I_i$ belong to pairwise distinct intervals in \mathcal{I} . For every $f : [s] \rightarrow \{0, 1\}$, we have

$$\Pr[\forall i, W_{\mathcal{I}}(x_i, y) = f(i) \mid y \in I_k] = \frac{1}{2^s},$$

where y is a random variable taking values uniformly in $[0, 1]$. It follows that for every graph H on s vertices,

$$\Pr[G(x_1, \dots, x_s, W_{\mathcal{I}}) = H \mid \forall i, x_i \in I_{k_i}] = \frac{1}{2^{\binom{s}{2}}}, \quad (2.1)$$

where x_1, \dots, x_s are now i.i.d. random variables taking values uniformly in $[0, 1]$, and k_1, k_2, \dots, k_s are distinct natural numbers.

We translate (2.1) into a lower bound on (conditional) entropy of transversal-uniform graphons. First we need a simple lemma.

Lemma 2.1. *Let $W_{\mathcal{I}}$ be a transversal-uniform graphon, and $\phi : [n] \rightarrow [0, 1]$ be a uniformly random map. For every $\rho : [n] \rightarrow \mathbb{N}$, we have*

$$\text{Ent}(G(\phi(1), \dots, \phi(n), W_{\mathcal{I}}) \mid \pi_{\mathcal{I}} \circ \phi = \rho) \geq \binom{|\text{Im}(\rho)|}{2}.$$

Proof. Pick a set of representatives $K \subseteq [n]$ so that $\rho|_K : K \rightarrow \text{Im}(\rho)$ is a bijection. Equation (2.1) implies that for every graph H with $V(H) = K$,

$$\Pr[G(\phi(1), \dots, \phi(n), W_{\mathcal{I}})[K] = H \mid \pi_{\mathcal{I}} \circ \phi = \rho] = \frac{1}{2^{\binom{|\text{Im}(\rho)|}{2}}}.$$

Therefore,

$$\text{Ent}(G(\phi(1), \dots, \phi(n), W_{\mathcal{I}}) \mid \pi_{\mathcal{I}} \circ \phi = \rho) \geq \text{Ent}(G(\phi(1), \dots, \phi(n), W_{\mathcal{I}})[K] \mid \pi_{\mathcal{I}} \circ \phi = \rho) = \binom{|\text{Im}(\rho)|}{2}.$$

□

In the proof of Theorem 1.1 below we will make use of the following well-known inequality about conditional entropy. For discrete random variables X and Y ,

$$\text{Ent}(X | Y) := \sum_{y \in \text{supp}(Y)} \Pr[Y = y] \text{Ent}(X | Y = y) \leq \text{Ent}(X). \quad (2.2)$$

Proof of Theorem 1.1. For every positive integer k , define

$$g_k := \max \{ \{2^{k+5}\} \cup \{n \mid \alpha(n) > 2^{-2k-9}\} \}.$$

The numbers g_k are well-defined, as the condition $\lim_{n \rightarrow \infty} \alpha(n) = 0$ implies that the set $\{n \mid \alpha(n) > 2^{-2k-9}\}$ is finite. Define the sums $G_k := \sum_{i=1}^k g_i$, and set $\beta_i = \frac{1}{g_i 2^k}$ for all the g_k indices $i \in (G_{k-1}, G_k]$. Let $\mathcal{I} = \{I_i\}_{i \in \mathbb{N}}$ be a partition of $[0, 1]$ into intervals with $|I_i| = \beta_i$, and let $W_{\mathcal{I}}$ be the corresponding transversal-uniform graphon.

Consider a sufficiently large $n \in \mathbb{N}$, and let $k \in \mathbb{N}$ be chosen to be maximum so that $2^{k+4} \leq n$ and $\alpha(n) \leq 2^{-2k-7}$. We have $n < 2^{k+5}$ or $\alpha(n) > 2^{-2k-9}$. Therefore $n \leq g_k$ by the definition of g_k . Let $\phi : [n] \rightarrow [0, 1]$ be random and uniform. By Lemma 2.1, for any fixed $\rho : [n] \rightarrow \mathbb{N}$, we have

$$\text{Ent}(G(\phi(1), \dots, \phi(n), W_{\mathcal{I}}) | \pi_{\mathcal{I}} \circ \phi = \rho) \geq \binom{|\text{Im}(\rho)|}{2}.$$

Thus

$$\text{Ent}(G(n, W_{\mathcal{I}})) \geq \text{Ent}(G(n, W_{\mathcal{I}}) | \pi_{\mathcal{I}} \circ \phi) \geq \Pr[|\text{Im}(\pi_{\mathcal{I}} \circ \phi)| \geq n2^{-k-2}] \binom{n2^{-k-2}}{2}. \quad (2.3)$$

Define the random variable $X := |\text{Im}(\pi_{\mathcal{I}} \circ \phi) \cap (G_{k-1}, G_k]| \leq |\text{Im}(\pi_{\mathcal{I}} \circ \phi)|$. We have

$$\mathbb{E}[X] = \sum_{i \in (G_{k-1}, G_k]} \Pr[\phi^{-1}(I_i) \neq \emptyset] = \sum_{i \in (G_{k-1}, G_k]} (1 - (1 - \beta_i)^n) = g_k \left(1 - \left(1 - \frac{1}{g_k 2^k} \right)^n \right) \geq n2^{-k-1},$$

where we used the fact that $g_k 2^k \geq 2n$ and that $(1 - x)^n \leq 1 - nx + n^2 x^2 \leq 1 - nx/2$ for $x \in [0, 1/2n]$. As the events $\phi^{-1}(I_i) \neq \emptyset$ and $\phi^{-1}(I_j) \neq \emptyset$ are negatively correlated for $i \neq j$, we have $\text{Var}[X] \leq \mathbb{E}[X]$. Hence by Chebyshev's inequality

$$\begin{aligned} \Pr[|\text{Im}(\pi_{\mathcal{I}} \circ \phi)| \geq n2^{-k-2}] &\geq \Pr[X \geq n2^{-k-2}] \geq 1 - \Pr\left[|X - \mathbb{E}[X]| \geq \frac{\mathbb{E}[X]}{2}\right] \\ &\geq 1 - \frac{4\text{Var}[X]}{\mathbb{E}[X]^2} \geq 1 - \frac{4}{n2^{-k-2}} \geq \frac{1}{2}. \end{aligned}$$

Substituting in (2.3) we obtain

$$\text{Ent}(G(n, W_{\mathcal{I}})) \geq \frac{1}{2} \binom{n2^{-k-2}}{2} \geq n^2 2^{-2k-7} \geq \alpha(n) n^2,$$

as desired. \square

3. PROOF OF THEOREM 1.2

In [LS] Lovász and Szegedy obtained a combinatorial characterization of random-free graph properties. To state this result it is convenient to distinguish between bipartite graphs and bigraphs. A *bipartite* graph is a graph (V, E) whose node set has a partition into two classes such that all edges connect nodes in different classes. A *bigraph* is a triple (U_1, U_2, E) where U_1 and U_2 are finite sets and $E \subseteq U_1 \times U_2$. So a bipartite graph becomes a bigraph if we fix a bipartition and specify which bipartition class is first and second. On the other hand, if $F = (V, E)$ is a graph, then (V, V, E') is an associated bigraph, where $E' = \{(x, y) : xy \in E\}$.

If $G = (V, E)$ is a graph, then an induced sub-bigraph of G is determined by two (not necessarily disjoint) subsets $S, T \subseteq V$, and its edge set consists of those pairs $(x, y) \in S \times T$ for which $xy \in E$ (so this is an induced subgraph of the bigraph associated with G).

For a bigraph $H = (U_1, U_2, E)$ and a graphon W , analogous to the definition of the induced density of a graph in a graphon, we define

$$p^b(H; W) = \mathbb{E} \left[\prod_{\substack{u \in U_1, v \in U_2 \\ uv \in E}} W(x_u, y_v) \prod_{\substack{u \in U_1, v \in U_2 \\ uv \in (U_1 \times U_2) \setminus E}} (1 - W(x_u, y_v)) \right],$$

where $\{x_u\}_{u \in U_1}, \{y_v\}_{v \in U_2}$ are independent random variables taking values in $[0, 1]$ uniformly. Now we are ready to state Lovász and Szegedy's characterization of random-free graph properties.

Theorem 3.1. [LS] *A graph property \mathcal{Q} is random-free if and only if there exists a bigraph H such that $p^b(H; W) = 0$ for all $W \in \overline{\mathcal{Q}}$.*

The following lemma is due to Alon, Fischer, and Newman (See [AFN07, Lemma 1.6]).

Lemma 3.2. [AFN07] *Let k be a fixed integer and let $\delta > 0$ be a small real. For every graph G , either there exists stepfunction graphon W' with $r \leq \left(\frac{k}{\delta}\right)^{O(k)}$ steps such that $\delta_1(W_G, W') \leq \delta$, or for every bigraph H on k vertices $p^b(H; G) \geq \left(\frac{\delta}{k}\right)^{O(k^2)}$.*

Every random-free graphon W can be approximated arbitrarily well in the δ_1 distance with W_G for some graph G , and furthermore, for every fixed H , the function $p^b(H, \cdot)$ is continuous in the δ_1 distance. Thus Lemma 3.2 can be generalized to random-free graphons.

Corollary 3.3. *Let k be a fixed integer and let $\delta > 0$ be a small real. For every random-free graphon W , either there exists a stepfunction graphon W' with $r \leq \left(\frac{k}{\delta}\right)^{O(k)}$ steps such that $\delta_1(W, W') \leq \delta$, or for every bigraph H on k vertices $p^b(H; G) \geq \left(\frac{\delta}{k}\right)^{O(k^2)}$.*

Next we will prove two simple lemmas about entropy.

Lemma 3.4. *Let μ_1 and μ_2 be two discrete probabilistic distributions on a finite set Ω . Then*

$$|\text{Ent}(\mu_1) - \text{Ent}(\mu_2)| \leq |\Omega| h \left(\frac{\|\mu_1 - \mu_2\|_1}{|\Omega|} \right).$$

Proof. Define $0 \log 0 := \lim_{x \rightarrow 0} x \log x = 0$. By taking the derivative with respect to x , for fixed d we see that $(x + d) \log(x + d) - x \log x$ is monotone for $0 \leq x \leq 1 - d$. Therefore, for $x_1, x_2 \in [0, 1]$ we have

$$|x_2 \log x_2 - x_1 \log x_1| \leq \max\{-|x_2 - x_1| \log |x_2 - x_1|, -(1 - |x_2 - x_1|) \log(1 - |x_2 - x_1|)\} \leq h(|x_2 - x_1|).$$

Thus

$$\begin{aligned} |\text{Ent}(\mu_1) - \text{Ent}(\mu_2)| &= \left| \sum_{x \in \Omega} \mu_1(x) \log \mu_1(x) - \mu_2(x) \log \mu_2(x) \right| \\ &\leq \sum_{x \in \Omega} h(|\mu_1(x) - \mu_2(x)|) \leq |\Omega| h \left(\frac{\|\mu_1 - \mu_2\|_1}{|\Omega|} \right), \end{aligned}$$

where the last inequality is by concavity of the binary entropy function h . \square

Lemma 3.5. *Let W_1 and W_2 be two graphons, and let μ_1 and μ_2 be the probability distributions on \mathcal{L}_n induced by $G(n, W_1)$ and $G(n, W_2)$, respectively. Then*

$$\|\mu_1 - \mu_2\|_1 \leq n^2 \delta_1(W_1, W_2).$$

Proof. Let x_1, \dots, x_n be i.i.d. uniform random variables with values in $[0, 1]$. Note

$$\begin{aligned} \|\mu_1 - \mu_2\|_1 &\leq \Pr[G(x_1, \dots, x_n, W_1) \neq G(x_1, \dots, x_n, W_2)] \\ &\leq \mathbb{E} \left[\sum_{i \neq j} |W_1(x_i, x_j) - W_2(x_i, x_j)| \right] \leq n^2 \|W_1 - W_2\|_1. \end{aligned}$$

\square

Proof of Theorem 1.2. Since \mathcal{Q} is random-free, by Theorem 3.1, there exists a bigraph H such that $p^b(H; W) = 0$ for all $W \in \overline{\mathcal{Q}}$. Applying Corollary 3.3 with $\delta = 1/n^5$ shows that there exists a stepfunction graphon W' with $n^{O(1)}$ steps satisfying $\|W - W'\|_1 \leq \delta$. Then since $|\mathcal{L}_n| \leq 2^{n^2}$, Lemmas 3.4 and 3.5 imply

$$\begin{aligned} |\text{Ent}(G(n, W')) - \text{Ent}(G(n, W))| &\leq 2^{n^2} h\left(\frac{n^2\delta}{2^{n^2}}\right) \\ &= -2^{n^2} \left(\frac{n^2\delta}{2^{n^2}} \log\left(\frac{n^2\delta}{2^{n^2}}\right) + \left(1 - \frac{n^2\delta}{2^{n^2}}\right) \log\left(1 - \frac{n^2\delta}{2^{n^2}}\right) \right) \\ &\leq n^4\delta + n^2\delta(-2\log n - \log \delta) + 2^{n^2} \cdot 2 \frac{n^2\delta}{2^{n^2}} = o(1). \end{aligned}$$

Since W' is random-free and it has $n^{O(1)}$ steps, $|\text{supp}(G(n, W'))| = n^{O(n)}$. Consequently $\text{Ent}(G(n, W')) = O(n \log n)$. \square

4. CONCLUDING REMARKS

1. Note that if W is a random-free stepfunction, then $\text{Ent}(G(n, W)) = O(n)$. In [Jan] it is conjectured that the converse is also true. That is $\text{Ent}(G(n, W)) = O(n)$ if and only if W is equivalent to a random-free stepfunction. The following simple example disproves this conjecture.

Let μ be the probability distribution on \mathbb{N} defined by $\mu(\{i\}) = 2^{-i}$. Consider the random variable $X = (X_1, \dots, X_n) \in \mathbb{N}^n$ where X_i are i.i.d. random variables with distribution μ . We have $\text{Ent}(X_i) = \sum_{i=1}^{\infty} 2^{-i} i = 2$. Hence $\text{Ent}(X) \leq \sum \text{Ent}(X_i) = 2n$.

Partition $[0, 1]$ into intervals $\{I_i\}_{i=1}^{\infty}$ where $|I_i| = 2^{-i}$. Let W be the graphon that is constant 1 on $\cup_{i=1}^{\infty} I_i \times I_i$ and 0 everywhere else. Note that

$$\text{Ent}(G(n, W)) \leq \text{Ent}(X) \leq 2n.$$

Therefore $G(n, W)$ has linear entropy.

It remains to verify that W is not equivalent to a stepfunction. This follows immediately from the fact that W has infinite rank as a kernel. It can also be verified in a more combinatorial way: A *homogenous set* of vertices in a graph H is a set of vertices which are either all pairwise adjacent to each other, or all pairwise non-adjacent. If W is equivalent to a step-function with k steps, then every $H \in \text{supp}(G(n, W))$ clearly contains a homogenous set of size at least n/k . On the other hand, if $H \in \mathcal{L}_{n^2}$ is a disjoint union of n complete graphs on n vertices, then the largest homogenous set in H has size n , but $H \in \text{supp}(G(n^2, W))$ by construction.

2. Theorem 1.2 shows that when W is a limit of a random-free property, then the entropy of $G(n, W)$ is small. However, the support of $G(n, W)$ can be comparatively large. For every $\epsilon > 0$, we construct examples for which $\log(|\text{supp}(G(n, W))|) = \Omega(n^{2-\epsilon})$. Note that Theorem 1.2 implies that $G(n, W)$ is far from being uniform on the support in these examples, as the entropy of a uniform random variable with support of size $2^{\Omega(n^{2-\epsilon})}$ is $\Omega(n^{2-\epsilon})$.

Let us now describe the construction. Let \mathcal{Q} be the set of graphs that do not contain $K_{t,t}$ as a subgraph. Partition $[0, 1]$ into intervals $\{S_i\}_{i=1}^{\infty}$ with non-zero lengths, and let $\{H_i\}_{i=1}^{\infty}$ be an enumeration of graphs in \mathcal{Q} . Define W to be the graphon that is 0 on $S_i \times S_j$ for $i \neq j$, and is equivalent to W_{H_i} (scaled properly) on $S_i \times S_i$. By construction $p(H; W) > 0$ if $H \in \mathcal{Q}$. Thus $|\text{supp}(G(n, W))| \geq |\mathcal{Q}_n|$. Since there exists $K_{t,t}$ -free graphs with $n^{2-2/t}$ edges (See e.g. [Bol78, p. 316, Thm. VI.2.10]), we have $|\mathcal{Q}_n| \geq 2^{n^{2-2/t}}$.

It remains to show that W is a limit of graphs in some random-free property. Unfortunately, $W \notin \overline{\mathcal{Q}}$. We construct a larger random-free property \mathcal{Q}' so that $W \in \overline{\mathcal{Q}'}$ as follows.

Fix a bigraph B , so that the corresponding graph contains $K_{t,t}$ as a subgraph and is connected. Suppose further that no two vertices of B have the same neighborhood. Note that such a bigraph trivially exists. For example, one can take $B = (V_1 \cup U_1, V_2 \cup U_2, E)$ so that V_1, U_1, V_2, U_2 are disjoint sets of size t , every vertex of V_1 is joined to every vertex of V_2 , and the edges between V_1 and U_2 , as well as the edges between U_1 and V_2 , form a matching of size t . Let $\mathcal{Q}' \supseteq \mathcal{Q}$ be the set of graphs not containing B as an induced sub-bigraph. Then \mathcal{Q}' is random-free by Theorem 3.1, as $p^b(B, W') = 0$ for every $W' \in \overline{\mathcal{Q}'}$.

Let $r = |V(B)|$ and suppose that $G = G(x_1, x_2, \dots, x_r, W)$ contains B as an induced sub-bigraph. Then there exists i so that $x_1, x_2, \dots, x_r \in S_i$, as G is connected. It follows further that G is an induced subgraph of H_i , as no two vertices of G have the same neighborhood. Thus G contains no $K_{t,t}$ subgraph, contradicting our assumption that G contains B . We conclude that $\text{supp}(G(n, W)) \subseteq \mathcal{Q}'$ for every positive integer n . By [LS06, Lemma 2.6] the sequence $\{G(n, W)\}_{n=1}^\infty$ converges to W with probability one. Thus $W \in \overline{\mathcal{Q}'}$, as desired.

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